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Extension of Bernstein polynomials to infinite dimensional case

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Abstract

The purpose of this paper is to study some new concrete approximation processes for continuous vector-valued mappings defined on the infinite dimensional cube or on a subset of a real Hilbert space. In both cases these operators are modelled on classical Bernstein polynomials and represent a possible extension to an infinite dimensional setting.

The same idea is generalized to obtain from a given approximation process for function defined on a real interval a new approximation process for vector-valued mappings defined on subsets of a real Hilbert space.

MSC: 41A10; 41A36; 25E15

Keywords: Bernstein polynomials; Infinite dimension approximation

1. Introduction

The purpose of this paper is to define an explicit sequence of operators that is an approximation process for continuous vector-valued mappings $F : X \to E$, where X has "infinite dimension". More precisely we deal with two cases. The first is when X is the cube

$$C_\infty := [0, 1]^{\mathbb{N}^*},$$

with the canonical product topology, where $\mathbb{N}^*$ denotes the set $\mathbb{N} \setminus \{0\}$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. The other case we consider, is when X is an unbounded, closed subset of a real Hilbert space endowed with the weak topology.

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A first approximation process, we are going to construct, is modelled on the Bernstein polynomials. Later we shall give a generalization of this construction.

The Bernstein polynomials, for a continuous function $F \in \mathscr{C}(C_k)$, on the k-dimensional cube $C_k := [0, 1]^k$, are defined at $t = (t_1, \dots, t_k) \in C_k$, as

$$B_{n,k}(F)(t) := \sum_{\substack{j_1=0\\ \cdots \\ j_k=0}}^{n} F\left(\frac{j_1}{n}, \dots, \frac{j_k}{n}\right) \psi_{n,j_1}(t_1) \cdots \psi_{n,j_k}(t_k),$$

where

$$\psi_{n,j}(t) := \binom{n}{j} t^j (1-t)^{n-j}.$$

It is well known that the sequence $(B_{n,k})_{n \geqslant 1}$ realizes an approximation process on $\mathscr{C}(C_k)$ as specified by

Theorem 1.1. 1. *For any $F \in \mathscr{C}(C_k)$, $B_{n,k}(F) \to F$ uniformly on C_k as $n \to \infty$.*

2. *Let C_k endow with the distance $d(x, y) := \sum_{i=1}^{k} |x_i - y_i|$. If $F \in Lip_M(C_k)$, then $B_{n,k}(F) \in Lip_M(C_k)$.*[1]

3. *For any convex function $F \in \mathscr{C}(C_k)$, $B_{n,k}(F)$ is convex with respect to each variable.*

4. *For any $F \in \mathscr{C}(C_k)$, convex with respect to each variable and $n \geqslant 1$, it results $F \leqslant B_{n,k}(F)$.*

5. *For any $F \in \mathscr{C}(C_k)$, convex with respect to each variable and $n \geqslant 1$, it results $B_{n+1,k}(F) \leqslant B_{n,k}(F)$.*

We refer the interested reader to e.g. [1,2,5].

Our idea is simple. We link the index n to the dimension k of the cube where the operator $B_{n,k}$ samples the function, obtaining the operator $B_{n,n}$; in the C_∞ case, the nth operator acts sampling $F : C_\infty \to E$ on a n-dimensional cube.

In the next section we present the results, while the proofs are in Section 3. The last section is devoted to extend the idea to other operators.

2. Definitions and results

Let X be a Hausdorff space and E a normed space. We denote with $\mathscr{F}(X, E)$ and with $\mathscr{C}(X, E)$ respectively the space of all mappings $F : X \to E$ and its subspace containing only the continuous mappings.

Fix $g : X \to \mathbb{R}_+$, the symbol $\mathscr{F}(X, E, g)$ stands for the subspace of all mappings F belonging to $\mathscr{F}(X, E)$ such that F/g is bounded.

For every $n \geqslant 1$, we set

$$A_n := \{h = (h_j)_{j \geqslant 1} | h_j \in \mathbb{N},\ 0 \leqslant h_j \leqslant n \text{ for } j \leqslant n,\ h_j = 0 \text{ for } j > n\}.$$

In other words, $h \in A_n$ if and only if it has the form $h = (h_1, \dots, h_n, 0, 0, \dots)$ with $0 \leqslant h_j \leqslant n$ for every natural $j \in \{1, 2, \dots, n\}$.

[1] Let (X, d) be a metric space and E normed space. A function $f : X \to E$ belongs to $Lip_M(X)$, if $\|f(t) - f(\tau)\| \leqslant M d(t, \tau)$, for any $t, \tau \in X$.

2.1. C_∞ case

As we have mentioned in the introduction, the topology in $C_\infty = [0,1]^{\mathbb{N}^*}$ is the canonical product one; every point $t \in C_\infty$ is identified with the sequence $(t_j)_{j\geqslant 1}$.

Let $n \geqslant 1$ be natural number, $h = (h_j)_{j\geqslant 1} \in A_n$. Define the function

$$\varphi_{n,h}(t) := \binom{n}{h_1}\cdots\binom{n}{h_n} t_1^{h_1}(1-t_1)^{n-h_1}\cdots t_n^{h_n}(1-t_n)^{n-h_n}, \tag{2.1}$$

for every $t \in C_\infty$. Notice that φ has the form $\varphi_{n,h}(t) = \psi_{n,h_1}(t_1)\cdots\psi_{n,h_n}(t_n)$.

For every $n \geqslant 1$, $F : C_\infty \to E$ and $t \in C_\infty$, we define

$$L_n^1(F)(t) := \sum_{h\in A_n} F\left(\frac{h}{n}\right)\varphi_{n,h}(t),$$

or, explicitly,

$$L_n^1(F)(t) = \sum_{\substack{h_1=0\\ \cdots \\ h_n=0}}^{n} F\left(\frac{h_1}{n},\ldots,\frac{h_n}{n},0,0,\ldots\right)\binom{n}{h_1}t_1^{h_1}(1-t_1)^{n-h_1} \times\cdots\binom{n}{h_n}t_n^{h_n}(1-t_n)^{n-h_n}.$$

In Section 3 we shall prove the following approximation result:

Theorem 2.1. *For any $F \in \mathscr{C}(C_\infty, E)$, the convergence*

$$L_n^1(F) \to F \quad \text{as } n \to \infty$$

holds uniformly on C_∞.

2.2. Hilbert case

Let H be an infinite dimension separable real Hilbert space. With $(a_j)_{j\geqslant 1}$ we denote a Hilbert base of H, so that the points $t \in H$ are represented by $t = \sum_{j=1}^{\infty} t_j a_j$. A well-known fact says that H is isometrically isomorphic to the Hilbert space $\ell^2 := \{(t_n)_{n\geqslant 1} | \sum_{n=1}^{\infty} |t_n|^2 < \infty\}$. Therefore, we shall use the identification $H = \ell^2$.

We set

$$\Gamma := \{t \in H \mid 0 \leqslant t_i \leqslant 1\}.$$

The definition of $\varphi_{n,k}$ in (2.1) is still valid for $t \in \Gamma$, hence for every $n \geqslant 1$, $F : \Gamma \to E$ and $t \in \Gamma$, we define

$$L_n^2(F)(t) := \sum_{h\in A_n} F\left(\frac{h}{n}\right)\varphi_{n,h}(t), \tag{2.2}$$

or, equivalently:

$$L_n^2(F)(t) = \sum_{\substack{h_1=0 \\ \cdots \\ h_n=0}}^{n} F\left(\frac{h_1}{n}, \ldots, \frac{h_n}{n}, 0, 0, \ldots\right) \binom{n}{h_1} t_1^{h_1}(1-t_1)^{n-h_1} \times \cdots \binom{n}{h_n} t_n^{h_n}(1-t_n)^{n-h_n}.$$

We remind the following definitions

Definition 2.2. Let X be a convex subset of a Banach space Y.

(1) The symbol $UCB(X, E)$ stands for the subspace of $\mathscr{F}(X, E)$ of all the uniformly continuous and bounded mappings. For $F \in UCB(X, E)$, we define, as usual, its *modulus of continuity*, as

$$\omega(F, \delta) := \sup\{\|F(u) - F(t)\| \,|u, t \in X, \|u - t\| \leqslant \delta\} \quad (\delta > 0).$$

(2) We say that $F: X \to E$ is *weak-to-norm continuous* if it is continuous from X equipped with the weak topology $\sigma(Y, Y')$ in Y, into E with the norm topology. By $\mathscr{K}(X, E)$ we denote the space of all weak-to-norm continuous mappings from X into E. We set $\mathscr{K}(X, E, g) := \mathscr{K}(X, E) \cap \mathscr{F}(X, E, g)$.

The approximation results in the Hilbert case are as follows.

Theorem 2.3. *For any $F \in \mathscr{K}(\Gamma, E, 1 + \|\cdot\|^2)$, the convergence*

$$L_n^2(F)(t) \to F(t)$$

holds for any $t \in \Gamma$ and uniformly on relatively compact subsets of Γ.

Theorem 2.4. *For any $F \in UCB(\Gamma, E)$, we have $L_n^2(F) \to F$ (as $n \to \infty$), uniformly on relatively compact subsets of Γ. Moreover for any $t \in \Gamma$, $n \geqslant 1$ and $\delta > 0$, there holds the estimate*

$$\left\| L_n^2(F)(t) - F(t) \right\| \leqslant \omega(F, \delta) \left[1 + \delta^{-2} \left(\sum_{j>n} t_j^2 + \sum_{j=1}^{n} \frac{t_j - t_j^2}{n} \right) \right],$$

therefore, in particular

$$\left\| L_n^2(F)(t) - F(t) \right\| \leqslant 2\omega \left(F, \sqrt{\sum_{j>n} t_j^2 + \sum_{j=1}^{n} \frac{t_j - t_j^2}{n}} \right) \leqslant 2\omega \left(F, \sqrt{\sum_{j>n} t_j^2 + \frac{\|t\|}{\sqrt{n}} + \frac{\|t\|^2}{n}} \right).$$

These operators L_n^2 satisfy the following preserving properties.

Proposition 2.5. 1. *If* $F \in Lip_M(\Gamma)$, *then* $L_n^2(F) \in Lip_{\sqrt{n}M}(\Gamma)$ *for any* $n \geqslant 1$.

2. *If* $F \in \mathscr{C}(\Gamma, \mathbb{R})$ *is convex, then for any* $n \geqslant 1$, $L_n^2(F)$ *is convex with respect to each variable*.

Thus, the analogues of the properties 1, 2 and 3 of Theorem 1.1 are in some sense inherited from L_n^2.

Let E be an ordered space. The following question arises. What happens to properties 4 and 5? They fail even in the case $E = \mathbb{R}$. We shall prove this claim in the next section finding a counterexample.

3. Proofs

Before proving the statements of the previous section, we recall the following definitions (cf. [4]). For any function $g \in \mathscr{F}(X, \mathbb{R})$ and any vector $\mathbf{v} \in E$, with $g \otimes \mathbf{v}$ we denote the function belonging to $\mathscr{F}(X, E)$ defined as

$$(g \otimes \mathbf{v})(t) := g(t)\mathbf{v} \quad \text{for any } \ t \in X.$$

Definition 3.1. Let S be a linear operator on $\mathscr{F}(X, \mathbb{R})$. A linear operator L on $\mathscr{F}(X, E)$ is said to be *S-regular* if

$$L(g \otimes \mathbf{v}) = S(g) \otimes \mathbf{v} \quad \text{for all } \ g \in \mathscr{F}(X, \mathbb{R}) \quad \textit{and} \ \ \mathbf{v} \in E.$$

L is said *monotonically regular*, if it is S-regular for some positive linear operator on $\mathscr{F}(X, \mathbb{R})$.

Remark 3.2. The operators L_n^1 and L_n^2 are well defined on scalar functions as well as on vector-valued mappings and we shall use the same symbol for the operators acting on vector-valued mappings or on scalar functions. Moreover, it is easy to see that both operators are monotonically regular.

3.1. Proof of Theorem 2.1

Combining the results [1, Theorem 4.4.6] and [4, Theorem 9, p. 111] we obtain

Theorem 3.3. *Let X be a compact Hausdorff space, E a normed linear space, M a subset of $\mathscr{C}(X, \mathbb{R})$ which separates the points of X, $\mathbf{v} \in E \setminus \{0\}$ and L_n a sequence of monotonically regular operators of $\mathscr{C}(X, E)$. If*

$$L_n(\mathbf{h}) \to \mathbf{h} \quad \textit{uniformly on} \ \ X$$

for any $\mathbf{h} \in \{\mathbf{1v}\} \cup \{h^j\mathbf{v} |\ h \in M, \ \ j = 1, 2\}$, *then*

$$L_n(F) \to F \quad \textit{uniformly on} \ \ X$$

for any $F \in \mathscr{C}(X, E)$.

Since L_n^1 is monotonically regular and C_∞ is compact, we shall use Theorem 3.3 to prove our Theorem 2.1.

Proof of Theorem 2.1. For $j \geqslant 1$, let $\mathrm{pr}_j : C_\infty \to \mathbb{R}$ be the canonical projection: $\mathrm{pr}_j(t) = t_j$. Let $\mathbf{v} \in E$ be a non zero constant, since $M = \{\mathrm{pr}_j \,|\, j \geqslant 1\}$ separates the points of C_∞, it is sufficient to check the convergences on the test function: $\mathbf{1v}$, $\mathrm{pr}_j\,\mathbf{v}$ and $\mathrm{pr}_j^2\,\mathbf{v}$.

$$\begin{aligned} L_n^1(\mathbf{1v})(t) &= \sum_{h \in A_n} \mathbf{v}\varphi_{n,h}(t) \\ &= \mathbf{v} \sum_{h_1=0}^{n} \binom{n}{h_1} t_1^{h_1}(1-t_1)^{n-h_1} \cdots \sum_{h_n=0}^{n} \binom{n}{h_n} t_n^{h_n}(1-t_n)^{n-h_n} = \mathbf{v}. \end{aligned}$$

For $j > n$,

$$\begin{aligned} L_n^1(\mathrm{pr}_j\mathbf{v})(t) &= \sum_{h \in A_n} \mathrm{pr}_j\left(\frac{h}{n}\right)\mathbf{v}\varphi_{n,k}(t) = 0, \\ L_n^1(\mathrm{pr}_j^2\mathbf{v})(t) &= \sum_{h \in A_n} \mathrm{pr}_j^2\left(\frac{h}{n}\right)\mathbf{v}\varphi_{n,k}(t) = 0, \end{aligned}$$

while for $j \leqslant n$,

$$\begin{aligned} L_n^1(\mathrm{pr}_j\mathbf{v})(t) &= \sum_{h \in A_n} \mathrm{pr}_j\left(\frac{h}{n}\right)\mathbf{v}\varphi_{n,k}(t) = \mathbf{v} \sum_{h \in A_n} \frac{h_j}{n}\varphi_{n,k}(t) \\ &= \mathbf{v} \sum_{h_1=0}^{n} \binom{n}{h_1} t_1^{h_1}(1-t_1)^{n-h_1} \cdots \sum_{h_j=0}^{n} \frac{h_j}{n}\binom{n}{h_j} t_j^{h_j}(1-t_j)^{n-h_j} \\ &\quad \cdots \sum_{h_n=0}^{n} \binom{n}{h_n} t_n^{h_n}(1-t_n)^{n-h_n} = t_j\mathbf{v}, \end{aligned}$$

$$\begin{aligned} L_n^1(\mathrm{pr}_j^2\mathbf{v})(t) &= \mathbf{v} \sum_{h \in A_n} \frac{h_j^2}{n^2}\varphi_{n,k}(t) \\ &= \mathbf{v} \sum_{h_1=0}^{n} \binom{n}{h_1} t_1^{h_1}(1-t_1)^{n-h_1} \cdots \sum_{h_j=0}^{n} \frac{h_j^2}{n^2}\binom{n}{h_j} t_j^{h_j}(1-t_j)^{n-h_j} \\ &\quad \cdots \sum_{h_n=0}^{n} \binom{n}{h_n} t_n^{h_n}(1-t_n)^{n-h_n} = t_j^2\mathbf{v} + \frac{t_j - t_j^2}{n}\mathbf{v}. \end{aligned}$$

From these identities, we conclude the proof. □

3.2. Hilbert case: proofs

We begin recalling the definition (cf. [3,4])

Definition 3.4. Let $L : D(L) \to \mathscr{F}(X, E)$, $S : D(S) \to \mathscr{F}(X, \mathbb{R})$ be linear operators, with $D(L)$ and $D(S)$ subspaces of $\mathscr{F}(X, E)$ and $\mathscr{F}(X, \mathbb{R})$, respectively. L is said to be *dominated by* S if

$$\|F\| \in D(S) \quad \text{and} \quad \|L(F)(t)\| \leqslant S(\|F\|)(t)$$

for any $F \in D(L)$ and $t \in X$.

As already stated in Remark 3.2, the operators acting on vector-valued mappings and on scalar functions will be denoted with the same symbol L_n^2. Therefore, the operator $L_n^2 : \mathscr{F}(\Gamma, E) \to \mathscr{F}(\Gamma, E)$ is dominated by $L_n^2 : \mathscr{F}(\Gamma, \mathbb{R}) \to \mathscr{F}(\Gamma, \mathbb{R})$.

In order to prove Theorems 2.3 and 2.4, we shall use the results stated in [3], which, for the sake of completeness, we report below.

Theorem 3.5. *Let Y and E be normed spaces, X be a convex subset of Y, $K \subset X$ and for any $n \geqslant 1$, $L_n \colon D(L_n) \to \mathscr{F}(K, E)$ be a S_n-regular linear operator dominated by the positive linear operator $S_n \colon D(S_n) \to \mathscr{F}(K, \mathbb{R})$. We suppose that, for every $n \geqslant 1$, $UCB(X, E) \subset D(L_n)$, $UCB(X, \mathbb{R}) \subset D(S_n)$ and $\psi_t^2 := \|\cdot - t\|^2 \in D(S_n)$ for some (and hence for all) $t \in Y$. Then for each $F \in UCB(X, E)$, $t \in K$ and $\delta > 0$, one has*

$$\|L_n(F)(t) - F(t)\| \leqslant \|F(t)\| \, |S_n(\mathbf{1})(t) - 1| + \omega(F, \delta)\left[S_n(\mathbf{1})(t) + \delta^{-2}\gamma_n^2(t)\right], \tag{3.3}$$

where $\gamma_n^2(t) := S_n(\psi_t^2)(t)$.

From Theorem 4.1 and Remarks 4.2 and 4.3 in [3], we deduce the following:

Theorem 3.6. *Let Y be a real reflexive Banach space, E normed space, X a convex subset of Y closed and unbounded or open, K a bounded, closed convex subset of X and $g : X \to \mathbb{R}$ satisfying the following conditions: g is strictly positive, strictly convex, Fréchet differentiable on K, $g'(K)$ is bounded in Y' and the function*

$$h(t, u) := g(u) - \left[g(t) + \langle g'(t), u - t\rangle\right],$$

is lower semicontinuous with respect to weak topology. Moreover, setting $B_n := g^{-1}([0, n])$, we require that $K \subset B_n$, B_n is bounded, $X \setminus B_n \neq \emptyset$ and

$$\lim_{\substack{\|t\| \to \infty \\ t \in X}} \frac{g(t)}{\|t\|} = +\infty.$$

For each $n \geqslant 1$, let $L_n \colon D(L_n) \to \mathscr{F}(K; E)$ be a S_n-regular linear operator dominated by the linear positive operator $S_n \colon D(S_n) \to \mathscr{F}(K, \mathbb{R})$, with $\mathscr{K}(X, E, g) \subset D(L_n)$, $\mathscr{K}(X, \mathbb{R}, g) \subset D(S_n)$ and $g, h \in D(S_n)$. If for every continuous linear functional $\phi \in Y'$, the convergences

$$S_n(\mathbf{1})(t) \to 1, \;\; S_n(\phi_{|X})(t) \to \phi(t) \quad \textit{and} \quad S_n(g)(t) \to g(t) \tag{3.4}$$

hold uniformly for $t \in K$, then for every $F \in \mathscr{K}(X; E, g)$ and $f \in \mathscr{K}(X, \mathbb{R}, g)$,

$$L_n(F)(t) \to F(t) \quad \textit{and} \quad S_n(f)(t) \to f(t) \quad \textit{uniformly for } t \in K.$$

In our case Y is the real separable Hilbert space H, X is the set Γ that results to be convex, unbounded and closed. In order to prove the pointwise convergence in Theorems 2.3 and 2.4 we have only to check the convergences in (3.4) and to evaluate the quantities involved in (3.3). The proof of the uniform convergence will need the following lemma.

Lemma 3.7. *Let $C \subset \ell^2$ be relatively compact. Then for any $\varepsilon > 0$, there exists an integer number $i = i(\varepsilon, C)$, such that for every $x \in C$, we have*

$$\sum_{j \geqslant i} x_j^2 < \varepsilon.$$

Proof. Suppose, contrary to our claim, that there exist $\varepsilon > 0$ and a sequence $(x^i)_{i \geqslant 1}$ in C, such that

$$\sqrt{\sum_{j \geqslant i} (x_j^i)^2} \geqslant \sqrt{\varepsilon}$$

for every $i \geqslant 1$. From the relatively compactness of C, there exists $\bar{x} \in \overline{C}$ such that (up to a subsequence), $x^i \to \bar{x}$ (as $i \to \infty$). Thus, we have

$$\sqrt{\varepsilon} \leqslant \sqrt{\sum_{j \geqslant i} (x_j^i)^2} \leqslant \sqrt{\sum_{j \geqslant i} (x_j^i - \bar{x}_j)^2} + \sqrt{\sum_{j \geqslant i} (\bar{x}_j)^2} \leqslant \left\| x^i - \bar{x} \right\| + \sqrt{\sum_{j \geqslant i} (\bar{x}_j)^2},$$

for every $i \geqslant 1$. Letting $i \to \infty$, we have a contradiction. □

Proof of Theorem 2.3. We begin fixing $A \subset \Gamma$ relatively compact and set K the compact convex hull of A. Setting $g(u) := 1 + \|u\|^2$, we have that the function

$$h(t, u) = \|t\|^2 + \|u\|^2 - 2\langle t, u \rangle,$$

is lower semicontinuous for the weak topology. Choosing λ such that $K \subset g^{-1}([0, \lambda])$, we have that the hypotheses of Theorem 3.6 are satisfied.

Now, with the same computations of the proof of Theorem 2.1, we evaluate the convergences on the test functions.

We begin with

$$L_n^2(\mathbf{1})(t) = \sum_{h \in A_n} \varphi_{n,h}(t) = 1. \tag{3.5}$$

Let us denote with $(e_j)_{j \geqslant 1}$ the dual base of $(a_j)_{j \geqslant 1}$ (that is the base of the dual space H' such that $\langle e_i, a_j \rangle = \delta_{ij}$). For $j > n$,

$$L_n^2(e_j)(t) = \sum_{h \in A_n} e_j \left(\frac{h}{n} \right) \varphi_{n,k}(t) = 0,$$

$$L_n^2(e_j^2)(t) = \sum_{h \in A_n} e_j^2 \left(\frac{h}{n} \right) \varphi_{n,k}(t) = 0,$$

while for $j \leqslant n$,

$$L_n^2(e_j)(t) = \sum_{h \in A_n} e_j \left(\frac{h}{n}\right) \varphi_{n,k}(t) = t_j,$$
$$L_n^2(e_j^2)(t) = \sum_{h \in A_n} \frac{h_j^2}{n^2} \varphi_{n,k}(t) = t_j^2 + \frac{t_j - t_j^2}{n}.$$

Let $t \in H$ and $\phi \in H'$, representing them as $t = \sum_{j=1}^{\infty} t_j a_j$ and $\phi = \sum_{j=1}^{\infty} \phi_j e_j$, we have $\phi(t) = \sum_{j=1}^{\infty} \phi_j t_j$. Computing

$$L_n^2(\phi)(t) = L_n^2 \left(\sum_{j=1}^{\infty} \phi_j e_j \right)(t) = \sum_{j=1}^{\infty} \phi_j L_n^2(e_j)(t) = \sum_{j=1}^{n} \phi_j t_j,$$

we obtain the convergence of $L_n^2(\phi)$ to ϕ, uniformly on bounded subsets of Γ.

Noting that $\psi_t^2(u) = \|u\|^2 + \|t\|^2 - 2\langle t, u\rangle$, in order to conclude the proofs, we have to evaluate $L_n^2(\|\cdot\|^2)$ on relatively compact subsets. From identity

$$\|t\|^2 = \sum_{j=1}^{\infty} t_j^2 = \sum_{j=1}^{\infty} e_j^2(t),$$

we have

$$L_n^2(\|\cdot\|^2)(t) = \sum_{j=1}^{\infty} L_n^2(e_j^2)(t) = \sum_{j=1}^{n} \left(t_j^2 + \frac{t_j - t_j^2}{n} \right),$$

and hence

$$L_n^2(\|\cdot\|^2)(t) - \|t\|^2 = -\sum_{j>n} t_j^2 + \sum_{j=1}^{n} \frac{t_j}{n} - \frac{1}{n} \sum_{j=1}^{n} t_j^2, \tag{3.6}$$

$$L_n^2(\psi_t^2)(t) = \sum_{j>n} t_j^2 + \sum_{j=1}^{n} \frac{t_j}{n} - \frac{1}{n} \sum_{j=1}^{n} t_j^2. \tag{3.7}$$

For the second term in the right-hand side of (3.6) and (3.7), the following estimate holds

$$\sum_{j=1}^{n} \frac{t_j}{n} \leqslant \left(\sum_{j=1}^{n} \frac{1}{n^2} \right)^{1/2} \left(\sum_{j=1}^{n} {t_j}^2 \right)^{1/2} \leqslant \frac{1}{\sqrt{n}} \|t\| .$$

Thus, the last two terms in (3.6) and (3.7) decay to 0 uniformly on bounded subsets of Γ. Therefore, the estimates and the convergences hold pointwise as claimed in Theorem 2.3. The uniform convergences on K (and hence on A) follow from the uniform convergence of $\sum_{j>n} t_j^2$ to 0, and this is stated in the above Lemma (3.7). $\square$

Proof of Theorem 2.4. In order to prove the estimates in the statement of Theorem 2.4, taking into account Theorem 3.5, it is sufficient to compute $L_n^2(\mathbf{1})(t)$ and $L_n^2(\|\cdot - t\|^2)(t)$. These quantities are already computed in the proof of Theorem 2.3. Hence from (3.5) and (3.7), we obtain the stated estimates. □

Proof of Proposition 2.5. The preserving properties of Proposition 2.5, follow from the definition of L_n^2 and from Theorem 1.1. For instance, the inclusion $L_n^2(Lip_M(\Gamma)) \subset Lip_{\sqrt{n}M}(\Gamma)$ follows from 2 of Theorem 1.1 and the relation

$$\sum_{i=1}^{n} |t_i| \leqslant \sqrt{n} \left(\sum_{i=1}^{n} |t_i|^2 \right) \leqslant \sqrt{n}\, \|t\| \,. \tag{3.8}$$

See Proposition 4.3 for more general cases. □

When E is the real line, it remains to prove that the analogues of properties 4 and 5 of Theorem 1.1 fail for L_n^2. Indeed, it is enough to consider what happens with the functionals e_j, the base of H': for $j > n$, $L_n^2(e_j) = 0$ and $n \geqslant j$, $L_n^2(e_j) = e_j$. Thus, one can conjecture that the properties hold definitively, that is, for any $f \in \mathscr{K}(\Gamma, \mathbb{R}, g)$ convex, there exists an integer v such that, for $n \geqslant v$, $L_n^2(f) \geqslant f$ and $L_n^2(f) \geqslant L_{n+1}^2(f)$. Though, even this conjecture is doomed to fail. Indeed, let $\bar{f}$ be the function defined as $\bar{f} := \sum_{j \geqslant 1} \frac{e_j}{2^j}$. The function $\bar{f}$ is convex and belongs to $\mathscr{K}(\Gamma, \mathbb{R}, g)$. Computing

$$L_n^2(\bar{f})(t) = \sum_{j=1}^{n} \frac{1}{2^j} \left(t_j^2 + \frac{t_j - t_j^2}{n} \right),$$

and applying at $\bar{t} = (1, \ldots, 1, t_{n+1}, t_{n+2}, \ldots)$, we obtain

$$\bar{f}(\bar{t}) - L_n^2(\bar{f})(\bar{t}) = \sum_{j \geqslant n+1} \frac{t_j^2}{2^j} \geqslant 0,$$

$$L_{n+1}^2(\bar{f})(\bar{t}) - L_n^2(\bar{f})(\bar{t}) = \frac{1}{2^{n+1}} \left(t_{n+1}^2 + \frac{t_{n+1} - t_{n+1}^2}{n+1} \right) \geqslant 0,$$

that prove our claims.

4. A generalization

In this section we generalize the proposed scheme. We start with a generic sequence of positive linear operators, and as in Bernstein polynomials case, we obtain approximation processes for vector-valued mappings defined on subsets of an infinite dimensional Hilbert space.

Let E be a Banach space, I a Hausdorff space, $J \subset I$ and for $n \geqslant 1$, and $t \in J$, $\mu_n(\cdot; t)$ a probability measure on σ-algebra of all Borel subset of I. With $L^1(I, E, \mu_n(\cdot; t))$, we denote the subspace of $\mathscr{F}(I, E)$ of all $\mu_n(\cdot; t)$-integrable functions. We consider the linear integral operator $L_{n,1} : L^1(I, E, \mu_n(\cdot; \cdot)) \to \mathscr{F}(J, E)$, defined as

$$L_{n,1}(f)(t) := \int_I f(u)\, d\mu_n(u; t).$$

From the measure $\mu_n(\cdot, t)$, we define for $n \geqslant 1$, $k \geqslant 1$ and $t = (t_1, \dots, t_k) \in J^k$ the product measure $\mu_{n,k}(\cdot; t) := \bigotimes_{i=1}^k \mu_n(\cdot; t_i)$, and then we consider the associated integral operator:

$$L_{n,k}(f)(t) := \int_{I^k} f(u)\, d\mu_{n,k}(u; t) = \int_{I^k} f(u_1, \dots, u_k)\, d\mu_n(u_1; t_1) \otimes \cdots \otimes d\mu_n(u_k; t_k),$$

for $t = (t_1, \dots, t_k) \in J^k$, and $f \in L^1(I^k, E, \mu_{n,k}(\cdot; t))$.

We fix $s = (s_i)_{i \geqslant 1} \in I^{\mathbb{N}^*}$. For $f : I^{\mathbb{N}^*} \to E$, the symbol f_k stands for the function $f_k : I^k \to E$ defined as $f_k(t_1, \dots, t_k) := f(t_1, \dots, t_k, s_{k+1}, s_{k+2}, \dots)$. In the other direction, for $f : I^k \to E$, the symbol $\tilde{f}$ denotes the function $\tilde{f} : I^{\mathbb{N}^*} \to E$, defined as $\tilde{f}(t) := f(t_1, \dots, t_k)$. Finally, for $f : I^{\mathbb{N}^*} \to E$ such that $f_n \in L^1(I^n, E, d\mu_{n,n}(\cdot; t))$, for any $t \in J^{\mathbb{N}^*}$, we define

$$L_n(f) := (L_{n,n}(f_n))\tilde{}.$$

It is immediate to check that L_n is a monotonically regular operator.

One can hope that some property of $L_{n,1}$ are inherited from L_n. For instance, choosing $L_{n,1} = B_{n,1}$, the Bernstein operators, $s = 0$, it results $L_n^2(f) = (B_{n,n}(f_n))\tilde{}$, for $f : \Gamma \to E$. If we define L_n^2 with a generic $s \in \Gamma$,

$$L_n^2(f)(t) := \sum_{h \in A_n} f\left(\frac{h_1}{n}, \dots, \frac{h_n}{n}, s_{n+1}, s_{n+2}, \dots\right) \varphi_{n,h}(t),$$

then this variation is not essential. Indeed, Theorems 2.3, 2.4 and their proofs are the same, and with a small change of the function $\bar{f}$, one can show that analogue properties of 4 and 5 of Theorem 1.1 do not hold.

Theorem 4.1. *In the same setting of Subsection* 2.2 *and with the above notation, let* $I = J$ *be a real interval with* $0 \in I$, $\Gamma' := \{t \in H |\ t_i \in I\}$, *and fix* $s = (s_i)_{i \geqslant 1} \in \Gamma'$. *We assume that* $e_2 \in L^1(I, \mathbb{R}, \mu_n(\cdot; t))$ *for every* $n \geqslant 1$ *and* $t \in J$, $L_{n,1}(e_1) = e_1$ *and* $L_{n,1}(e_2) = e_2 + e_2 o(1) + e_1 o(\frac{1}{\sqrt{n}}) + o(\frac{1}{n})$.

1. *If* $F \in \mathscr{K}(\Gamma', E, 1 + \|\cdot\|^2)$, *or* $F \in UCB(\Gamma', E)$, *then*

$$L_n(F) \to F$$

uniformly on relatively compact subsets of Γ'.

2. *If* $L_{n,1}(Lip_1(I)) \subset Lip_1(I)$, *then* $L_n(Lip_M(\Gamma')) \subset Lip_{\sqrt{n}M}(\Gamma')$.

We note that the conditions of Theorem 4.1 are satisfied by many operators, e.g. Szász–Mirakjan operators, Baskakov operators, Post–Widder operators.

Remark 4.2. In the assumption $L_{n,1}(e_2) = e_2 + e_2 o(1) + e_1 o(\frac{1}{\sqrt{n}}) + o(\frac{1}{n})$, the last term cannot be substituted with the weaker condition $O(\frac{1}{n})$. Indeed, let $L_{n,1}$ be the Gauss–Weierstrass operators, defined for $t \in \mathbb{R}$ and $f \in \mathscr{C}(\mathbb{R}, \exp(e_2))$, as

$$L_{n,1}(f)(t) := \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} f(u) e^{-n(u-t)^2} du.$$

It results $L_{n,1}(\mathbf{1}) = \mathbf{1}$, $L_{n,1}(e_1) = e_1$, $L_{n,1}(e_2) = e_2 + \frac{1}{2n}$ and $L_{n,1}$ approximates uniformly on bounded sets the functions belonging to $\mathscr{C}(\mathbb{R}, \exp(e_2))$ (see [6]).

Choosing $s_i = 0$, with same notation as before, we get $L_n(\|\cdot\|^2)(t) = \sum_{i=1}^n t_i^2 + 1/2$, which converges to $\|t\|^2 + 1/2$. Therefore, we cannot conclude that L_n is an approximation process for functions belonging to $\mathscr{K}(H, \mathbb{R}, 1 + \|\cdot\|^2)$.

In order to prove the preserving property 2, we give the following result.

Proposition 4.3. *Let (I, d) be metric space. Consider the metric space I^k endowed with the distance $d_k(t, \tau) := \sum_{i=1}^k d(t_i, \tau_i)$. If $L_{n,1}(Lip_1(I)) \subset Lip_1(I)$, then for any $k \geqslant 1$* we have $L_{n,k}(Lip_1(I^k)) \subset Lip_1(I^k)$.

Proof. We shall argue by induction on k. For $k = 1$, the property holds by hypothesis. We assume that it is true for $k - 1$. Let $f \in Lip_1(I^k)$, $t, \tau \in I^{k-1}$ and $t_k, \tau_k \in I$. Using the integral nature of the operators $L_{n,k}$, one gets

$$\begin{aligned} &L_{n,k}(f)(t, t_k) - L_{n,k}(f)(\tau, \tau_k) \\ &\quad = \int_I \left[L_{n,k-1}(f(\cdot, u_k))(t) - L_{n,k-1}(f(\cdot, u_k))(\tau)\right] d\mu_n(u_k; t_k) \\ &\qquad + \int_{I^{k-1}} \left[L_{n,1}(f(u, \cdot))(t_k) - L_{n,1}(f(u, \cdot))(\tau_k)\right] d\mu_{n,k-1}(u; t). \end{aligned}$$

Thus, since $f_{|I^{k-1}} \in Lip_1(I^{k-1})$, we obtain

$$\begin{aligned} &\left\|L_{n,k}(f)(t, t_k) - L_{n,k}(f)(\tau, \tau_k)\right\| \\ &\quad \leqslant \int_I d_{k-1}(t, \tau)\, d\mu_n(u_k; t_k) + \int_{I^{k-1}} d(t_k, \tau_k)\, d\mu_{n,k-1}(u; t) \\ &\quad = d_k((t, t_k), (\tau, \tau_k)), \end{aligned}$$

which allows us to conclude the proof of the proposition. □

Proof of Theorem 4.1. The proof of the approximation property of Theorem 4.1, using Theorems 3.5 and 3.6, is the same of Theorems 2.3 and 2.4.

In the setting of Theorem 4.1, the inclusion $L_n(Lip_M(\Gamma')) \subset Lip_{\sqrt{n}M}(\Gamma')$ is now immediate. Indeed, if $f \in Lip_1(\Gamma')$, then also its restriction f_k belongs to $Lip_1(I^k)$, for every $k \geqslant 1$. Hence, $L_{n,n}(f_n) \in Lip_1(I^n)$, and from inequality (3.8), we get the thesis. □

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